PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Mo.

2709. Proposed by E. V. HUNTINGTON, Cambridge, Mass.

The following problem was suggested to the proposer by a professor of biology, who has found the result useful in certain problems concerning the equilibrium of chemical reactions.

Starting with $\mu(c_1 + y - x)(c_2 - x)\cdots(c_n - x) = \lambda(b_1 + x)(b_2 + x)\cdots(b_m + x)$, where $\mu c_1 c_2 c_3 \cdots c_n = \lambda b_1 b_2 b_3 \cdots b_m$ (all the letters being positive), find the limit of x/y as y approaches zero; and show that for small values of y, the value of x/y is always less than this limit.

2710. Proposed by ROGER E. MOORE, The University of Wisconsin.

If $a_k^{(r)}$ denotes the kth term of an arithmetic progression of order r; and c_k denotes the kth binomial coefficient in the expansion of $(a - b)^n$ (n being a positive integer), show that

$$S \equiv \sum_{k=1}^{n+1} c_k a_k^{(r)} = 0, \text{ if } n > r.$$

2711. Proposed by PAUL CAPRON, U. S. Naval Academy.

Show that the curves (a) $a^8 y_1^2 = x^4(a^2 - x^2)^3$, (b) $a^8 y_2^2 = x^8(a^2 - x^2)$ bound ten areas, of which two are each $(a^2/4)(\frac{1}{4}\pi - \frac{1}{3})$ and the remaining eight are each $a^2/24$.

2712. Proposed by WILLIAM HOOVER, Columbus, Ohio.

Given the conic $ax^2 + 2hxy + by^2 - 2x = 0$. Find the locus on which lie the four points of intersection of pairs of tangents to the conic from a pair of points on the x-axis equidistant from the origin.

2713. Proposed by G. PAASWELL, New York City.

In the design of gravity retaining walls the following relation exists,

$$\frac{k \cos (\phi' + a + b)}{\tan^2 b \cos b} - \frac{m \cos \phi' \sec a}{\tan^2 b} = 1,$$

where k, m, ϕ' are constants, a and b are the angles formed by the vertical with (1) the diagonal of the section running from the lower left of the section to the upper right, (2) the inside (right) face of the wall. The left face of the wall is vertical. Solve the equation for a, either exactly or by a good approximation. The area of the wall is given by $A = (\tan a + \tan b)/2$, the height of the wall is taken as unity. With the first equation given, determine a value of either a or b which will minimize this area, *i. e.*, what relation must exist between a and b to give the most economical section of wall?

2714. Proposed by H. R. HOWARD, University of St. Francis Xavier's College, Nova Scotia.

A shuffled pack of $2(p + q)$ cards contains $2p$ honors. Show that the chance of securing exactly half the honors in taking half the pack is $[F(p, q)]^2 \div F(2p, 2q)$, where $F(p, q)$ denotes the number of different sets of p cards which can be selected from $(p + q)$ cards.

Show also that if one honor is removed from the pack, the chance is not thereby affected. Is this true for the chance of getting any other assigned number of honors?

2715. Proposed by H. R. KINGSTON, University of Manitoba.

A', B', C' are points on the sides BC, CA, AB, respectively, of the triangle ABC, and AA', BB', CC' are concurrent in O. X, Y, Z are the three collinear points in which, by Desargues' theorem, the corresponding sides of the triangles ABC and $A'B'C'$ intersect. If A'', B'', C'' are the vertices of the triangle formed by the lines AX, BY, CZ, show that AA'', BB'', CC'' are concurrent.

2716. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

To a passenger in a train moving at the rate of 40 miles per hour, the rain appears to be rushing downward and towards him at an angle of 20 degrees with the horizontal. If the rain is

actually falling in a vertical direction, show that the velocity of the raindrops in feet per second is 21.35.

2717. Proposed by ENOS W. WITMER, Sophomore in Franklin and Marshall College.

Determine the integral values of m and n for which the equation $x^4 + mx^2y^2 + ny^4 = z^2$ has non-trivial solutions. [Carmichael's *Diophantine Analysis*, Prob. 13, p. 53.]

The following problems in volumes XX to XXIII still remain unsolved:
Algebra: numbers 406, 430, 461;
Geometry: numbers 446, 470, 472, 478, 494, 501, 510, 519, 523;
Calculus: numbers 339, 340, 342, 348, 353, 360, 385, 411, 415, 425, 429, 432, 434, 436, 440;
Mechanics: numbers 277, 279, 285, 287, 291, 308, 309, 313, 315, 322, 328, 335, 343, 344, 348, 350, 351, 356, 357;
Number Theory: numbers 189, 190, 191, 192, 200, 205, 231, 232, 234, 239, 245, 247, 260, 261, 263, 270, 271, 273, 274, 275.
The editors will be glad to receive solutions of any of these unsolved problems.

SOLUTIONS OF PROBLEMS.

Note. 1. Florence P. Lewis sent in a solution of 486 and Albert Babbitt a solution of 493, Algebra, after selection for publication had been made and sent to the Editor-in-Chief.

2. The following correction should be made: On page 24, of the January, 1918, number of the MONTHLY, 16th line from bottom, for $(n^2 + 1) - 4n$, read $(n + 1)^2 - 4n$. EDITORS.

427 (Calculus). Proposed by ROGER S. JOHNSON, Adelbert College, Cleveland, O.

Of all ellipses cirumscribed about a given parallelogram, the minimum (maximum) with regard to area has as conjugate diameters the diagonals of the parallelogram.

II. SOLUTION BY O. D. KELLOGG, University of Missouri.

I venture to add my solution to those given in the January number of the MONTHLY because it illustrates the fruitfulness of the notion "shear," a simple transformation which should doubtless find more use in elementary mathematics.[1] The coördinate axes having any position in the plane, the transformation $x = x'$, $y = y' - ax'$ defines a shear. The following are invariants: area, ellipse, conjugacy, parallelism.

Suppose E' and E'' are two ellipses circumscribed about the parallelogram P, the former having the diagonals of P as conjugate diameters. A shear may be found which carries E' over into a circle C', and consequently P into a square R', since its diagonals are conjugate diameters of a circle. Using the letters to denote areas, we have $P/E' = R'/C'$.

A second shear may be found which will carry E'' into a circle C'', and consequently P into a rectangle R'', and we have $P/E'' = R''/C''$.

But a square has an area whose ratio to that of the circumscribed circle is greater than that of any other rectangle. Hence $R'/C' > R''/C''$, and the above equations yield $E'' > E'$, so that E' is the circumscribed ellipse of *minimum* area.

438 (Calculus). Proposed by PAUL CAPRON, U. S. Naval Academy.

Find the locus of the equation

$$y^6 - 3(a^2 - x^2)y^4 - 2ax^2y^3 + 3(a^2 - x^2)^2y^2 - 6ax^2(a^2 - x^2)y + a^2x^4 - (a^2 - x^2)^3 = 0,$$

first showing that it can be reduced to the form

$$y = kx^n \pm (a^2 - x^2)^m,$$

and finding the points of maximum abscissa, of maximum ordinate, and of inflection.

---

[1] In fact the idea is used in Young and Morgan's *Elementary Analysis*, pages 132, 288, and 289.